Problem Session, 2 part 1

26 p56, hints for 20,21,22,23 pp81-82

Problem Session (Chapter 3)

26 p 56 $L = \{x \in \mathbb{R} \mid x > 0\}$

$a \oplus b = ab \qquad a \otimes b = a^{\log b}$

$\log b = \ln b = \log_e b$

$e \approx 2{,}718\ldots$

$a = e^{\log a}$

$a \otimes b = a^{\log b} = \left(e^{\log a}\right)^{\log b} = e^{\log a \cdot \log b}$

If L with these operations is a ring, then this ring is commutative ⑨

Ring axioms

Closure and associativity ①, ②, ⑥, ⑦ are straightforward

Commutativity ③ is implied by the commutativity of multiplication of real numbers.

④ $0_L = 1 \qquad a \oplus 1 = a \cdot 1 = a$

⑤ $a \oplus x = 0_L \qquad \ominus a = \frac{1}{a}$

⑧ Distributive law

$a \otimes (b \oplus c) = a \otimes (bc) = e^{\log a \cdot \log(bc)} = e^{\log a (\log b + \log c)}$

$= e^{\log a \log b + \log a \log c} = e^{\log a \cdot \log b} \cdot e^{\log a \log c}$

$= (a \otimes b) \oplus (a \otimes c)$ is satisfied

— L is a commutative ring

Identity $1_L = e$

$$a \otimes e = e^{\log a \cdot \overset{''1}{\log e}} = a$$

- commutative ring with identity

Integral domain $\quad 1_L = e \neq 1 = 0_L$

$$a \otimes b = 0_L$$

$$e^{\log a \cdot \log b} = 1$$

$$\log a \cdot \log b = 0$$

$$\log a = 0 \quad \text{or} \quad \log b = 0$$

$$a = 1 = 0_L \quad \text{or} \quad b = 1 = 0_L$$

- yes, L is an integral domain
(11) is satisfied

Field - yes (12) is satisfied

$a \otimes x = 1_L$ is solved by $x = e^{1/\log a}$, $\log a \neq 0$ because $a \neq 0_L$

Same problem from the point of view of Section 3.3

- Similar to 20, 21, 22, 23 on pp 81-82

The field L is isomorphic to the field $\mathbb{R}$ of all real numbers.

Prop L with the operations $\oplus$, $\odot$ is isomorphic to the field $\mathbb{R}$ of real numbers

Pf

$$\mathbb{R} \longrightarrow L = \{x \in \mathbb{R} \mid x > 0\}$$
$$x \longmapsto e^x$$

- one-to-one map between sets

logarithm is the inverse

$$x+y \longmapsto e^{x+y} = e^x e^y = e^x \oplus e^y$$

$$xy \longmapsto e^{xy} = e^{\log e^x \cdot \log e^y} = e^x \otimes e^y$$